Max-Min Energy Theory for the Time Optimal Control of Rotating Rigid Body

Ojika, Takeo; Nakazumi, Akira

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Max-Min Energy Theory for the Time Optimal Control of Rotating Rigid Body

Gifu Univ. Takeo Ojika (小鹿 丈夫)
Genesis Tech. Inc. Akira Nakazumi (中住 晃)

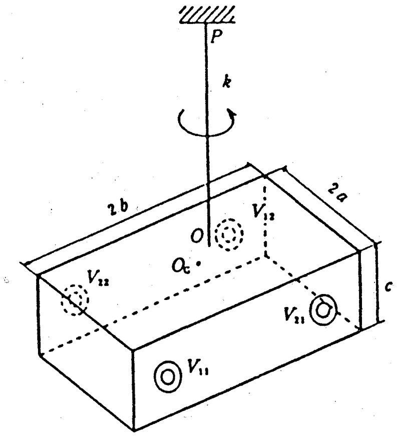
In this paper, we first formulate the equation of the motion of a rigid body suspended from a fixed point by a light wire and as an external force, uniform wind is blowing to the rigid body from some direction, which is then allowed to rotate in a horizontal plane. The equation is a nonlinear double integral plant. Secondly in order to control the rigid body we propose a practical control method by the bang-bang control strategy and then several properties of the problem are discussed from the energetic point of view, termed here as max-min energy theory, by which the switching point can be obtained analytically without applying the maximum principle by Pontryagin. Algorithms to transfer the rigid body from an initial state to the origin in the minimum time or in a finite number of switchings, and then to hold it at the origin are also given. Lastly several numerical examples are solved to show the effectivenesses of the present methods.

1. Introduction

From an engineering point of view, it is very important to control or to suppress the mechanical vibrations as can often be seen in crane systems [9,10].

In Section 2, we first consider the motion of a rigid body which consists of a rectangular prism. As shown in Fig. 1.1, the rigid body is suspended from the fixed point P by a light wire whose torsional factor is k. Taking practical applications into account, it is also assumed that uniform wind is blowing to the body from some direction. The torsional system of equation of the rigid body around a fixed point O which is slightly perturbed from the true center of gravity O_G can be expressed by a second-order nonlinear ordinary differential equation. Several properties of the system of equation are discussed from the energetic point of view, and it is shown that the system of equation has periodic solutions [4].

To control the motion of the rigid body, a power system is necessary. In Section 3, we propose a practical air or water jet control



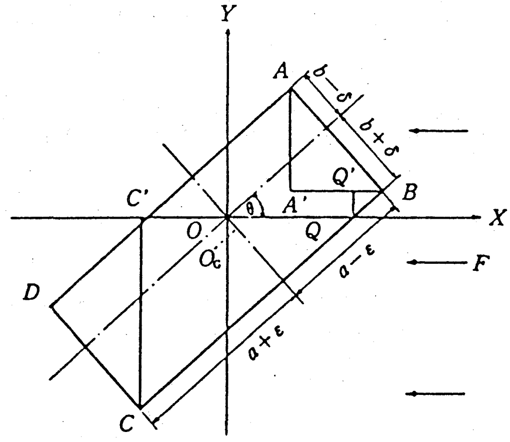


Fig.1.1. Rotating rigid body.   Fig.2.1. Rotating rigid body in wind.

system, and then we formulate a time optimal control problem which transfers the rigid body from an initial state to a final state (the origin) in the minimum time and after then holds it at the origin. It is interesting to show that the conservation of energy principle also holds for the time optimal control problem [5,6].

In section 4, several properties of the present control system of equations are discussed. According to the energy theory, the optimal control without switching is obtained easily without using the maximum principle of Pontryagin [1,8]. Secondly, the direction and the switching point of one switching control can be easily obtained analytically from an energy theory, termed here as max-min energy theory.

By the max-min energy theory, the time optimal control problem can ultimately be reduced to a set of two point boundary value problems, which can easily be solved by the initial value adjusting method with interval decomposition proposed by Ojika et. al. [13-17]. The outline of the method is given in Section 5.

In Section 6, the time optimal control problem is solved numerically, and the effectivenesses of the present methods are also shown.

2. Dynamical Model of a Rotating Rigid Body

2.1 Polar Moment of Inertia

As shown in Fig. 2.1, consider the rectangular rigid body with a base $2a \times 2b$ and a height of c whose mass is M. Let the center of gravity O_G be the origin. Then the moment of inertia about any point O whose coordinate is given by (ε, δ), termed here perturbed centroid, is obtained as follow [2,4,11,12]:

$$J = M\{a^2+b^2+3(\varepsilon^2+\delta^2)\}/3. \tag{2.1}$$

2.2 Torsional System of Equation

Suppose now that the rigid body is suspended horizontally from a fixed point P by a light wire and the anothor end is held fixed to the perturbed centroid O of the rigid body. As an external force, uniform wind whose pressure per unit area is F is blowing to the rigid body along the X axis (see Fig.2.1). Then the rigid body is allowed to rotate in a horizontal plane.

Note here that, in the subsequent discussions, the perturbed centroid O is taken to be the origin.

Let θ be the inclination of the rigid body to the X axis at a certain instant and consider first the torque T produced by wind pressure. From Fig. 2.1, the torque T about O is obtained to be [4]

$$T = -2cF(\varepsilon \sin\theta + \delta \cos\theta)(a \sin\theta + b \cos\theta). \tag{2.2}$$

It is well-known that the sum of the mass moment of the rigid body and the twist moment of the wire is equal to the external torque T, and we finally have the following torque equation about the perturbed centroid O:

$$J\ddot{\theta} + k\theta + 2cF(\varepsilon \sin\theta + \delta \cos\theta)(a \sin\theta + b \cos\theta) = 0, \quad k \neq 0 \tag{2.3}$$

which denotes the second-order nonlinear differential equation, where $\cdot\cdot$ denotes d^2/dt^2, and we here after call (2.3) as the perturbed rotating equation [3,4].

2.3 Conservation of Energy

It is interesting to show now that the conservation of energy principle holds for (2.3). In fact, since $\ddot{\theta} = \dot{\theta}(d\dot{\theta}/d\theta)$, the equation (2.3) becomes

$$\dot{\theta}(d\dot{\theta}/d\theta) + \omega^2\theta + \tau(\varepsilon \sin\theta + \delta \cos\theta)(a \sin\theta + b \cos\theta) = 0, \tag{2.4}$$

where

$$\omega = \sqrt{k/J}, \qquad \tau = 2cF/J, \tag{2.5}$$

and $\omega \neq 0$ is called the natural frequency. By integrating (2.4) with respect to θ, we have

$$E(0,\Theta) = (\dot{\theta}^2+\omega^2\theta^2)/2 + \tau[2(a\varepsilon + b\delta)\theta$$
$$+(b\delta - a\varepsilon)\sin 2\theta - (a\delta + b\varepsilon)\cos 2\theta]/4 \tag{2.6}$$

for all t, where $E(0,\Theta)$, $\Theta = (\theta, \dot{\theta})$ is a constant and denotes the total energy of the system (refer also to (3.8) in Section 3). In addition, it is easily seen that $E(0,\Theta) = E(0,\Theta')$, where $\Theta' = (\theta, -\dot{\theta})$.

The above discussions can be summarized in the form of a theorem.

Theorem 2.1. If the initial condition, Θ_0, of the perturbed rotating system of equation (2.3) is given, then the unique total energy $E(0,\Theta_0)$ is given by (2.6) and is constant independently to the time t, which shows that the system is energy conservative. Moreover, the energy constant contour of $E(0,\Theta)$ is symmetric with respect to θ-axis.

3. Time Optimal Control of Rigid Body

3.1. Control System of Equations

Let us first relabel θ and $\dot{\theta}$ as x_1 and x_2, respectively, called the state variable $x(t)=(x_1(t), x_2(t))$. Then, (2.3) can be rewritten in the form ofthe first-order system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\omega^2 x_1 - f(x_1), \end{aligned} \tag{3.1a}$$

where

$$f(x_1) = \tau(\varepsilon \sin x_1 + \delta \cos x_1)(a \sin x_1 + b \cos x_1). \tag{3.1b}$$

In order to transfer the rigid body at a given initial state to a final state against wind pressure, it is obvious that a power system, termed as control u(t), is necessary. For that, taking practical applications into account, we introduce an air or a water jet system with two sets of nozzles which are connected to an air or a water compressor with enough capacity via pipes, as shown in Fig.1.1. Here the set of nozzles V_{11} and V_{12}, or V_{21} and V_{22} works at the same time and can generate control torque about the pertubed centroid O. Moreover air or water pressure is assumed to be adjustable by a pressure control valve. Then (3.1) can be formulated to the following optimal control system of equations:

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= x_{10}, \\ \dot{x}_2 &= -\omega^2 x_1 - f(x_1) + Ku, & x_2(0) &= x_{20}, \end{aligned} \tag{3.2}$$

where $x(0)=(x_{10}, x_{20})$ is the given initial state at $t=0$, $u(t)$ is the control variable, and $K\ (>0)$ denotes the magnitude of air or water pressure set by the pressure control valve and is called a control gain. Let us here consider the time optimal control problem whose dynamical system of equations is given by (3.2), and suppose that the control variable u is constrained in magnitude by the relation

$$|u(t)| \leq 1 \tag{3.3}$$

for all t. In general, the magnitude constraint comes from physical limitation on the torque which is generated by air or water jet under some given pressure. However, as will be seen in the subsequent discussions, it can be considered that $u(t)$ simply shows the direction of control and takes +1 or -1.

Then the time optimal control problem is defined as follows: given the dynamical system (3.2) with the constraint (3.3) and a given gain constant K, find the optimal control $u^*(t)$ which minimizes the transfer time t_f from the initial state $x(0)=(x_{10}, x_{20})$ to the final state (0, 0)[1,8].

For the n-th order linear ordinary differential equations with a control u, it has been proved that the maximum number of switchings is at most n-1, if the control system of equations is completely controllable [1,8]. However, for nonlinear time optimal control problems, there is in general no guarantee that the maximum number of switchings is at most n-1. In the following, we will discuss from the energetic point of view that, under appropriate conditions, it also holds for the present nonlinear problem.

3.2 Conservation of Energy of the Time Optimal Control System

In Subsection 2.3, it has been shown that the conservation of energy principle holds for the system of equation (2.3). Let us show that the principle also holds for the system of equation (3.2). For that, define the following new coordinate system defined by

$$y_1(t) = x_1(t) - v(t), \qquad y_2(t) = x_2(t), \tag{3.4}$$

where

$$v(t) = (K/\omega^2)u(t), \tag{3.5}$$

and v is termed here as normalized control. Substituting (3.4) and (3.5) into (3.2) yields

$$\begin{aligned} \dot{y}_1 &= y_2, & y_1(0)&=x_{10}-v(0), \\ \dot{y}_2 &= -\omega^2 y_1 - f(y_1+v), & y_2(0)&=x_{20}. \end{aligned} \tag{3.6}$$

As seen in Subsection 2.3, we also have a constant energy $E(v,y)$, $y=(y_1, y_2)$ for the new system of equation (3.6):

$$\begin{aligned} E(v,y)=&(y_2{}^2+\omega^2 y_1{}^2)/2+\tau\,[2(a\varepsilon+b\delta)(y_1+v) \\ &+(b\delta-a\varepsilon)\sin 2(y_1+v)-(a\delta+b\varepsilon)\cos 2(y_1+v)]/4, \end{aligned} \tag{3.7}$$

for the time interval such that v is constant. Substituting (3.4) and

(3.5) into (3.7), we can rewrite $E(v,y)$ by use of the original coordinate system as

$$E(v,x)=[x_2{}^2+\omega^2(x_1-v)^2]/2+\tau[2(a\varepsilon+b\delta)x_1$$
$$+(b\delta-a\varepsilon)\sin 2x_1-(a\delta+b\varepsilon)\cos 2x_1]/4. \quad (3.8)$$

Similarly to Theorem 2.1, the above discussions for the optimal control system of equations (3.2) can be summarized as follow.

Theorem 3.2. If the initial state of the optimal control system of equations (3.2), say, $x(0)=(x_{10}, x_{20})$ and $Ku(0)=\omega^2 v(0)$ at $t=0$ are given, then the unique total energy $E(v,x(0))$ is given by (3.8) and is constant for the time interval such that v is constant, which shows that the system is energy conservative in the time interval. Moreover, the energy constant contour of $E(v,x)$ is symmetric with respect to x_1-axis.

As for the relation between the dynamical system (3.2) and the energy (3.8), the following theorem holds.

Theorem 3.3. Suppose that the energy of the time optimal control problemis given by (3.8) with a constant normalized control v defined by (3.5). Then the following relations hold:

$$\dot{x}_1 = \partial E(v,x)/\partial x_2, \quad (3.9a)$$

$$\dot{x}_2 = -\partial E(v,x)/\partial x_1, \quad (3.9b)$$

for all t.

The proof of the theorem is obvious. The theorem shows that properties of the dynamical system of equations (3.2) can be explained by examining those of the static energy equation (3.8).

4. Determination of Optimal Trajectory by Energy Theory

4.1. Control Without Switching

In the following, it is assumed that (i) the control $Ku(t)$ is kept constant for all t, and (ii) our object is to find the optimal control Ku and its trajectory which transfers the rigid body from a given initial state $x(0)=(x_{10}, x_{20})$ to the origin (0, 0) without switching the control u, and then to hold it to the origin. It is interesting to consider the conditions for transferring the rigid body from a given initial state to the origin without switching. For this, itis essential to discuss the relation between energy at the initial state and that of the origin.

From the energetic point of view, the following theorem holds.

Theorem 4.1. Suppose that the normalized control v^* given by (3.5) at the given initial state $x(0)=(x_{10}, x_{20})$, $x_{10}\neq 0$ at $t=0$ satisfies

$$v^*=\{2(x_{20}^2+\omega^2 x_{10}^2)+\tau[2(a\varepsilon+b\delta)x_{10}+(b\delta-a\varepsilon)\sin 2x_{10} \\ -(a\delta+b\varepsilon)(\cos 2x_{10}-1)]\}/(4\omega^2 x_{10}). \tag{4.1}$$

Then we have

$$E(v^*,x(0))=E(v^*,0), \tag{4.2a}$$

where

$$E(v^*,0)=\omega^2 v^{*2}/2-\tau(a\delta+b\varepsilon)/4, \tag{4.2b}$$

which shows that the energy at the initial state is equal to that at the origin.

Proof. From (3.8) with v^* and $x=x(0)$, we have

$$E(v^*,x(0))=\omega^2 v^{*2}/2+[x_{20}^2+\omega^2(x_{10}^2-2x_{10}v^*)]/2+\tau[2(a\varepsilon+b\delta)x_{10} \\ +(b\delta-a\varepsilon)\sin 2x_{10}-(a\delta+b\varepsilon)\cos 2x_{10}]/4 \\ =E(v^*,0). \tag{4.2c}$$

□

It is worth mentioning that v^* defined by (4.1) is a unique control gain $K^*=\omega^2|v^*|$ which can transfer the rigid body from an initial state to the origin without switching. However, depending on the initial state, it is not always possible to transfer to the origin with the normalized control v^*.

Further discussions will be seen in [7].

Theorem 4.2. Let the initial state be $x(0)=(0, x_{20})$, $x_{20}\neq 0$, and suppose that the normalized control v^* ($\neq 0$) is given by (4.1). Then $E(v^*,x(0))\neq E(v^*,0)$, and it is impossible to transfer the rigid body from the initial state to the origin without any switching.

Proof. Suppose that the energy $E(v^*,x(0))$ at the initial state is equal to $E(v^*,0)$ at the origin. Then, from (3.8), we have $x_{20}^2=0$, which contradicts with the assumption. □

Corollary 4.1. Suppose that all the conditions in Theorem 4.1 hold. Then the solution $x_1(t)$ of (3.2) with $x(0)=(x_{10}, x_{20})$, $x_{10}\neq 0$ and v^* ($\neq 0$) satisfies one of the following conditions:

(i) if $x_{10}>0$, then $x_1(t)\geqq 0$ for all t, (4.3a)

(ii) if $x_{10}<0$, then $x_1(t)\leqq 0$ for all t. (4.3b)

Proof. Since $\partial E(v^*,x)/\partial x_2=0$ at the origin, the energy curve $E(v^*,x)$ is tangent to the x_2-axis at the origin. From this fact and Theorem4.2, it is obvious that there is no energy constant curve $E(v^*,x(t))$for all t ($\geqq 0$) which intersects the x_2-axis at $(0,x_2)$, $x_2 \neq 0$. □

4.2. Control With Switching

We pointed out that properties of the time optimal control problem (3.2) can be explained by the energy $E(v,x)$ given by (3.8). The solution $(x_1(t),x_2(t))$ of (3.2) with the given initial state $x(0)=(x_{10}, x_{20})$, $|x_{10}|+|x_{20}| \neq 0$ at $t=0$ and the given normalized control v is periodic with period, say, $T(\geqq 0)$. Similarly to linear time optimal control problems, let us now consider the conditions for transferring the rigid body from a given initial state to the origin with one switching. For the present nonlinear ordinary differential equation given by (3.2), the following Lemma plays an important role in the subsequent discussions.

Lemma 4.1. Suppose that $y(t)=(y_1(t), y_2(t))$ and $z(t)=(z_1(t), z_2(t))$ be the solutions of (3.2) with the initial state $y(0)=z(0)=x(0)$, $|x_{10}|+|x_{20}| \neq 0$,and with the constant controls (i) $Ku(t)=\omega^2 v$, and (ii) $Ku(t)=-\omega^2 v$, $t\geqq 0$, $v\neq 0$, respectively. Then the following relations hold, respectively:

$$\text{(i)}\quad E(-v,y(t))=E(v,x(0))+2\omega^2 v y_1(t), \tag{4.4a}$$

$$\text{(ii)}\quad E(v,z(t))=E(-v,x(0))-2\omega^2 v z_1(t). \tag{4.4b}$$

Proof. From (3.8) with the solution $x=y(t)$ of (3.2) and $v=Ku/\omega^2$, we have

$$E(-v,y(t))=E(v,y(t))+2\omega^2 v y_1(t). \tag{4.5}$$

Since $E(v,y)$ is constant for all t by the conservation law of energy, $E(v,y(0))=E(v,y(t))$ holds for all t such that $v=Ku/\omega^2$ is constant, and hence we have (4.4a). Similarly, from (3.8) with $x=z(t)$ and $v=-Ku/\omega^2$, we easily obtain (4.4b). □

Analogously to linear time optimal control problems of double integral plant, it will be in general necessary to change the control Ku to -Ku at a switching point, say, $x(t_s)=(x_1(t_s),x_2(t_s))$, $t_s\geqq 0$ for transferring the rigid body from a given intial state to the origin. Here the instance t_s of switching the control is called switching time. For determining the switching point of the rigid body, the energies given by (4.4) will be of great help.

Theorem 4.3.1. Suppose that, for the given initial state $x(0)=(x_{10}, x_{20})$, the normalized control $v_+^*=K^*u^*/\omega^2$ satisfies the following condition:

$$x_{10}v_+^* \leqq [2(x_{20}^2+\omega^2 x_{10}^2)+c_0]/(4\omega^2), \tag{4.6a}$$

where

$$c_0=\tau\{2(a\varepsilon+b\delta)x_{10}+(b\delta-a\varepsilon)\sin(2x_{10})-(a\delta+b\varepsilon)[\cos(2x_{10})-1]\}. \tag{4.7a}$$

Then (i) the energy E satisfies the following inequality:

$$E(v_+^*,x(0)) \geqq E(-v_+^*,0), \tag{4.8a}$$

where

$$E(-v_+^*,0)=[2(\omega v_+^*)^2-\tau(a\delta+b\varepsilon)]/4, \tag{4.9}$$

(ii) there exists $y_1^*(t_{s1})$ at $t=t_{s1}$ such that

$$y_1^*(t_{s1})=[E(-v_+^*,0)-E(v_+^*,x(0))]/(2\omega^2 v_+^*). \tag{4.10a}$$

Moreover, (iii) if v_+^* also satisfies the following inequality:

$$y_1^*(t_{s1})v_+^* \leqq [-2\omega^2 y_1^*(t_{s1})^2-c_{s1}]/(4\omega^2), \tag{4.11a}$$

where

$$c_{s1}=\tau\{[2(a\varepsilon+b\delta)y_1^*(t_{s1})+(b\delta-a\varepsilon)\sin(2y_1^*(t_{s1}))-(a\delta+b\varepsilon)[\cos(2y_1^*(t_{s1}))-1]\}. \tag{4.12a}$$

Then d_1 defined by

$$d_1=-2\omega^2 y_1^*(t_{s1})[y_1^*(t_{s1})+2v_+^*]-c_{s1} \tag{4.13a}$$

satisfies

$$d_1 \geqq 0, \tag{4.13b}$$

and there exists $y_2^*(t_{s1})$ such that

$$y_2^*(t_{s1})=\pm\sqrt{d_1/2}. \tag{4.10b}$$

Proof. (i) From (3.8) and (4.6a), we easily have

$$E(v_+^*,x(0))-E(-v_+^*,0)= -\omega^2 x_{10}v_+^*+[2(x_{20}^2+\omega^2 x_{10}^2)+c_0]/4 \geqq 0. \tag{4.14}$$

(ii) From (4.4a) and (4.14), there exists $y_1^*(t_{s1})$ such that

$$E(-v_+^*,y^*(t_{s1}))=E(v_+^*,x(0))+2\omega^2 v_+^* y_1^*(t_{s1}) = E(-v_+^*,0) \tag{4.15}$$

holds at $t=t_{s1}$, which proves (4.10a).

(iii) Since $E(v_+^*,x(0))= E(v_+^*,y^*(t_{s1}))$, substituting $y_1^*(t_{s1})$ into (3.8) and taking the condition (4.11a) into account yield $d_1 \geqq 0$. From the relation $E(-v_+^*,y^*(t_{s1}))= E(-v_+^*,0)$ in (4.15), (4.10b) can be easily obtained. □

Similarly, consider (3.2) with the control $Ku=-\omega^2 v$. Then we have the following.

Theorem 4.3.2. Suppose that, for the given initial state x(0), the normalized control $v_-^*=-K^*u^*/\omega^2$ satisfies the following condition:

$$x_{10}v_-^* \geqq -[2(x_{20}^2+\omega^2 x_{10}^2)+c_0]/(4\omega^2). \tag{4.6b}$$

Then (i) the energy E satisfies the following inequality:

$$E(-v_-^*,x(0)) \geqq E(v_-^*,0), \tag{4.8b}$$

(ii) there exists $z_1^*(t_{s2})$ at $t=t_{s2}$ such that

$$z_1^*(t_{s2})=[E(-v_-^*,x(0))-E(v_-^*,0)]/(2\omega^2 v_-^*). \tag{4.10c}$$

Moreover, (iii) if v_-^* also satisfies the following inequality:

$$z_1^*(t_{s2})v_-^* \leqq [-2\omega^2 z_1^*(t_{s2})^2-c_{s2}]/(4\omega^2), \tag{4.11b}$$

where

$$c_{s2}=\tau\{2(a\varepsilon+b\delta)z_1^*(t_{s2})+(b\delta-a\varepsilon)\sin(2z_1^*(t_{s2}))$$
$$-(a\delta+b\varepsilon)[\cos(2z_1^*(t_{s2}))-1]\}. \tag{4.12b}$$

Then d_2 defined by

$$d_2=-2\omega^2 z_1^*(t_{s2})[z_1^*(t_{s2})+2v_-^*]-c_{s2} \tag{4.13c}$$

satisfies

$$d_2 \geqq 0, \tag{4.13d}$$

and there exists $z_2^*(t_{s2})$ such that

$$z_2^*(t_{s2})=\pm\sqrt{d_2/2}. \tag{4.10d}$$

It is worth mentioning that once the initial condition x_0 and control gain K are given, then all the possible candidates for the switching point can be analytically obtained in advance from (4.10).

Corollary 4.2. Suppose that the control gain $K=\bar{K}$ and its corresponding normalized control $v=\bar{v}$ satisfy the inequality:

$$\bar{K}=\omega^2|\bar{v}| \gg \max[\omega^2|x_{10}|,\omega^2|x_{20}|,|\tau a\varepsilon|,|\tau a\delta|,$$
$$|\tau b\varepsilon|,|\tau b\delta|], \tag{4.16}$$

then the following approximate relations hold:

$$\text{(i)}\quad \bar{x}_1(t_s) \fallingdotseq x_{10}/2, \tag{4.17a}$$

$$\text{(ii)}\quad \bar{x}_2(t_s) \fallingdotseq \pm\sqrt{-x_{10}\omega^2\bar{v}}. \tag{4.17b}$$

Proof. It is obvious that $E(\bar{v},\bar{x}(t))=E(\bar{v},x(0))$ for the solution of (3.2) with the initial state $x(0)$, $t\geqq 0$.

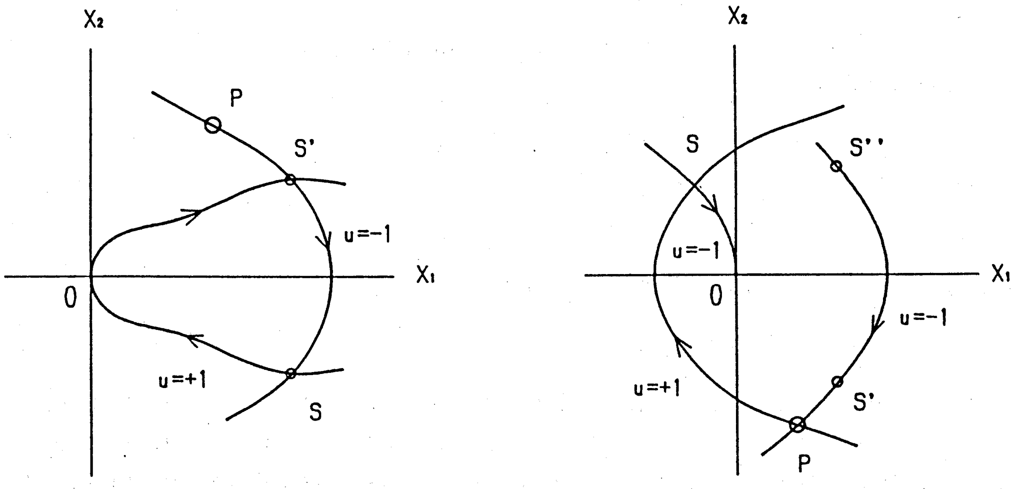
(i) Taking the assumption for $\bar{v}$ into account, from (4.10a) or (4.10c), we have

$$\bar{x}_1(t_s)= [E(-\bar{v},0)-E(\bar{v},x(0))]/(2\omega^2\bar{v})$$
$$\fallingdotseq x_{10}/2.$$

(ii) Substituting (4.17a) into (4.10b) or (4.10d), (4.17b) can be easily obtained. □

Remark 4.1. Similarly to linear double integral plants, this corollary guarantees that there exists the control gain K which can transfer the rigid body given by (3.2) from any initial state to the origin with one switching.

Let us now consider to determine the sign of $y_2^*(t_{s1})$ or $z_2^*(t_{s2})$ given by (4.10b) or (4.10d).



(a) $(x_2^*(t_s) < x_{20})$ (b) $(x_2^*(t_s) \geqq x_{20})$

Fig.4.1. Schematic diagram of switching points .

The trajectory of the solution $x^*(t)=(x_1^*(t),\ x_2^*(t))$ of (3.2) with a given control K^*u^* and the given initial state x_0 is shown in Fig.3.1(a), where the control sequence is taken to be {-1, +1}. In the figure, the initial state and the switching points are denoted by P and S, respectively.

Suppose now that x_{10} is positive and $x^*(t_s)=(x_1^*(t_s),\ +x_2^*(t_s))$, whose point is denoted by S' in the figure, is chosen to be a switching point. Then we have to reach the origin along the direction of the trajectory from the initial state P to the switching point S' and then against the direction of the trajectory from S' to the origin O. This fact means the motion of the rigid body for negative time. On the other hand, if we take $x^*(t_s)=(x_1^*(t_s),-x_2^*(t_s))$ as a switching point, whose point is denoted by S in the figure, then one can reach the origin along the trajectory PSO, provided the transition from control u=-1 to u=+1 occurs at the point S. It is also obvious that identical arguments holds for negative x_{10}.

We can summarize the above results in the following control law for the switching point S.

Control Law 4.1. For the given initial state $x(0)=(x_{10},x_{20})$ and the control K^*u^*, the sign of $x_2^*(t_s)$ at the switching point $x^*(t_s)$ is chosen so as to satisfy the following inequality:

$$\mathrm{sgn}\{x_{10}\}\cdot\mathrm{sgn}\{x_2^*(t_s)\} < 0, \tag{4.18}$$

where sgn denotes the signum function.

As shown in Fig.3.1(b) in which the switching points $x(t_s)$ are denoted by S' and S'', if the inequality $x_2^*(t_s)\geqq x_{20}$ holds for the given control K^*, then we have to reach from the initial state P to the switching points S' and S'' against the direction of the trajectory. This fact shows that the motion ofthe rigid body in the interval PS' and PS" is negative in time, and hence, in place of the control K^*u^*, it is necessary to adopt the control $-K^*u^*$.

Using identical arguments, we can conclude that if $x_{10}<0$ and the inequality $x_2^*(t_s)\leqq x_{20}$ holds for the given control K^*u^*, then it is necessary to adopt the control $-K^*u^*$.

From the above discussions, we can establish the following control law:

Control Law 4.2. For the given initial state $x(0)=(x_{10},x_{20})$ and the given control K^*u^*, suppose that $x_2^*(t_s)$ is computed from (4.10b) and the sign of $x_2^*(t_s)$ at the switching point $x^*(t_s)$ is chosen so as to satisfy Control Law 4.1. Then the direction of the time optimal control u at the initial time t=0 is given by

(i) if (a) $x_{10}\geqq 0$ and $x_2^*(t_s)-x_{20}<0$, or
(b) $x_{10}\leqq 0$ and $x_2^*(t_s)-x_{20}>0$, then $u=u^*$,

(ii) if (c) $x_{10}\geqq 0$ and $x_2^*(t_s)-x_{20}\geqq 0$, or
(d) $x_{10}\leqq 0$ and $x_2^*(t_s)-x_{20}\leqq 0$, then $u=-u^*$.

and, furthermore, $u=u^*$ or $u=-u^*$ is unique for the given control gain K^* (>0). Here the uniqueness of the control K^*u^* or $-K^*u^*$ can be easily shown from that of the energy constant curve E defined by (3.8) with $v=v^*$ or $v=-v^*$.

Remark 4.2. The above conditions (i) and (ii) show the conditions for (i) maximizing and (ii) minimizing the energy $E(v^*,x(0))$ with respect to the sign of the given normalized control v^* under the given initial state $x(0)$, respectively and hence the present theory is termed here as max-min energy theory.

Assuming now that x_{10} of the given initial state x_0 is positive, consider here the relation between x_{20} and $x_2^*(t_s)$ of the switching point $x^*(t_s)$, where $x_2^*(t_s^*)$ is computed from (4.10b) or (4.10d), and satisfies Control Law 4.1.

Theorem 4.4. Suppose that for the given initial condition $x(0)$ and $\hat{K}u=\hat{K}^*u^*$ $(=\omega^2\hat{v}^*)$, the following equality holds:

$$x^*(t_s)-x(0)=0, \tag{4.19}$$

where $x^*(t_s)=(x_1^*(t_s),x_2^*(t_s))$. Then the normalized control $\hat{v}^*$ coincides with $-v^*$ given by (4.1), which is the normalized time optimal control without switching, and moreover the energy $E(-\hat{v}^*,x(0))$ at the initial state coincides with $E(\hat{v}^*,0)$ at the origin, i.e.,

$$E(-\hat{v}^*,x(0))=E(\hat{v}^*,0). \tag{4.20}$$

Proof. Substituting $x^*(t_s)=x(0)$ into (4.10b) or (4.10d), and then squaring both sides, we have

$$\hat{v}^*=-[2(x_{20}^2+\omega^2x_{10}^2)+c_0]/(4\omega^2x_{10})=-v^*. \tag{4.21}$$

On the other hand, from (3.8), (4.9), and (4.21), we have

$$\begin{aligned}E(-\hat{v}^*,x(0))&=E(\hat{v}^*,x(0))\\&=E(\hat{v}^*,0)\\&=E(\hat{v}^*,0).\end{aligned}$$ □

4.3. Uniqueness of the Trajectories

Let us now consider the uniqueness of the trajectories $x^*(t)$ of (3.2) with the optimal control $K^*u^*=\omega^2v^*$ in the intervals $[0,t_s)$ and $(t_s,t_f]$.

The following theorem holds for the first interval.

Theorem 4.5. Suppose that the initial state $x(0)$ $(\neq 0)$ and the normalized control v^* which satisfies the Control Laws 4.1 and 4.2 are given. Then the solution $x^*(t)=(x_1^*(t),x_2^*(t))$, $t\geq 0$, of (3.2) which passes through the switching point $x^*(t_s^*)$ at $t=t_s^*$ is unique, where $t_s=t_s^*+nT_1$, $0\leq t_s^*<T_1$, $n=0,1,\cdots$, and T_1 (≥ 0) is the period of the solution curve of (3.2) with the given control $Ku=K^*u^*$.

Proof. From the assumptions, the switching point $x^*(t_s)$ which satisfies (4.10) and Control Law 4.2 exists. On the other hand, since the solution $x^*(t)$ of (3.2) with initial state $x(0)(\neq 0)$ and $K^*u^*=\omega^2 v^*$ is periodic and unique, it is obvious that there exists the minimum switching time t_s^* such that $t_s=t_s^*+nT_1$, $0\leq t_s^*<T_1$, $n=0,1,\cdots$. Hence the switching point $x^*(t_s^*)$ is unique. □

Similarly, the following theorem holds for the second interval.

Theorem 4.6. Suppose that we switch the control from $K^*u^*=\omega^2 v^*$ to $K^*u^*=-\omega^2 v^*$ at $t=t_s^*$, where the normalized control v^* satisfies the inequality (4.11). Then the solution of the initial value problem (3.2) with the initial state $(x_1^*(t_s^*),x_2^*(t_s^*))$ passes through the origin, say, at $t=t_f^*+nT_2$, $n=0,1,\cdots$, where $t_f^*\geq t_s^*$ and $T_2(\geq 0)$ is the period of the solution curve of (3.2) with the given control $-K^*u^*$. Moreover the trajectory $(x_1^*(t),\ x_2^*(t))$, $t\in(t_s^*,t_f^*]$ is unique.

Further detailed discussions will be given in [7].

4.4. Holding Control

At the practical crane systems, it is important to hold the weight cargos to the origin after it is transferred from an initial state to the origin. For the purpose, the following theorem is useful.

Theorem 4.7. Suppose that the rigid body is transferred from an initial state to the origin (0, 0) at, say, $t=t_f$ with a constant control $Ku=K_\theta^*u_\theta^*$ and then the control is switched to the following value:

$$K_\theta^*u_\theta^*(t)=\omega^2 v_\theta^* \quad \text{for all } t\geq t_f\ , \tag{4.22a}$$

where v_θ^* is given by

$$v_\theta^*=\tau b\delta/\omega^2, \tag{4.22b}$$

which is termed here as holding normalized control. Then the rigid body is held to the origin for all $t\geq t_f$ and its corresponding energy $E(v_\theta^*,0)$, termed here as holding energy, is given by

$$E(v_\theta^*,0)=[2(\tau b\delta/\omega)^2-\tau(a\delta+b\varepsilon)]/4. \tag{4.22c}$$

Proof. If acceleration $\dot{x}_2(=\ddot{\theta})$ is zero at the origin $(x_1(t_f),\ x_2(t_f))=(0,\ 0)$, then there is no force for the rigid body to leave from the origin and thus, from (3.2), we have

$$\dot{x}_2 = -\tau b \delta + Ku$$
$$= 0, \qquad (4.23)$$

which proves (4.22a). Substituting (4.22a) at the origin into (3.8), we easily have (4.22c). □

In order to hold the rigid body to the origin, this theorem shows that it is necessary to switch the control to $Ku = \tau b \delta$ at the origin and the energy of the control system is given by (4.22c). It is also interesting to point out that (i) the holding control is independent to the perturbation ε along X-axis, (ii) if the perturbation $\delta = 0$, the holding control is not necessary and the body stays at the origin even in the constant wind.

5. Algorithm for the Time Optimal Control

In the previous sections, we derived conditions for the time optimal control of the rigid body from the energetic point of view. However, since the ordinary differential equation (3.2) is nonlinear, it is still impossible to determine analytically the optimal switching time t_s^* and the terminal time t_f^*. Thus, let us now consider the algorithm for computing these values numerically.

5.1. Two Point Boundary Value Problems

Since (3.2) is a second-order differential equations, we can expect that there will be at most one switching at the unknown time t_s^*. Let us now introduce a new time $r \varepsilon [t_0, t_2]$ and time scale factors $\alpha_1(\geqq 0)$ and $\alpha_2(\geqq 0)$ defined by

(1) subinterval 1: $r = t/\alpha_1$, $r \varepsilon [t_0, t_1)$,

(2) subinterval 2: $r = t/\alpha_2$, $r \varepsilon (t_1, t_2]$, $t_0 < t_1 \leqq < t_2$. (5.1)

Here $t_0 = 0$, and t_1 and t_2 are the prescribed switching and terminal times, respectively, for the new time system r, and α_i, $i=1, 2$ is an unknown constant, termed as time scale factors. Substituting (5.1) into (3.2), we have for interval i:

$$dx_1^{(i)}/dr = \alpha_i x_2^{(i)},$$
$$dx_2^{(i)}/dr = \alpha_i [-\omega^2 x_1^{(i)} - f(x_1^{(i)}) + \omega^2 v^{(i)}], \qquad (5.2a)$$

where $v^{(1)} = -v^{(2)} = v^*$.

In addition, since α_i, $i=1, 2$, is assumed to be a constant, we have

$$d\alpha_i/dr = 0, \qquad i=1,\ 2. \tag{5.2b}$$

On the other hand, the given boundary conditions for (5.2) can be rewritten as

$$x_1^{(1)}(t_0)=x_{10}, \qquad x_2^{(1)}(t_0)=x_{20}, \tag{5.3a}$$

$$x_2^{(2)}(t_2)=0. \tag{5.3b}$$

Note here that since the variation of $x_1^{(2)}(t)$ in the neighborhood of the origin becomes very slow, which means that the convergence of the iterative computation also becomes slow, and hence the given terminal condition $x_1^{(2)}(0)=0$ is not adopted here. Instead, we apply the following boundary conditions:

$$x_2^{(1)}(t_1)=x_2^*(t_s^*), \tag{5.3c}$$

and

$$x_1^{(2)}(t_1)=x_1^*(t_s^*), \qquad x_2^{(2)}(t_1)=x_2^*(t_s^*), \tag{5.3d}$$

where $x(t_s^*)$ is obtained from (4.10).

5.2. Initial Value Adjusting Method

The ordinary differential equation (3.2) with the boundary conditions (5.3) constitutes two sets of two point boundary value problems which can be solved independently, and it is now possible to compute the time scale factor α_i, i=1,2 and hence the optimal solutions $x_1^{(i)}(t)^*$, $x_2^{(i)}(t)^*$, i=1, 2 of the present problem. For this purpose, initial value adjusting method (IVAM) with interval decomposition developed by Ojika [13] is very powerful.

For the ordinary differential equations (5.2), the initial states $x^{(1)}(t_0)=(x_{10},x_{20})$ and $x^{(2)}(t_1)=(x_1^*(t_s^*),x_2^*(t_s^*))$ are given, respectively. Hence, from (5.3b) and (5.3c), we define the following boundary conditions:

$$g(\alpha_1)=x_2^{(1)}(t_1)-x_2^*(t_s^*)=0, \tag{5.4a}$$

and

$$g(\alpha_2)=x_2^{(2)}(t_2)=0, \tag{5.4b}$$

for determining α_i in each subinterval.

Start now with prescribing the value of ${}^0\alpha_i$, and then solve a set of three dimensional initial value problems given by (5.2) with the initial state $x^{(1)}(t_0)$ or $x^{(2)}(t_1)$, respectively, and denote the solutions by $x=({}^kx_1^{(i)}(t),{}^kx_2^{(i)}(t))$ and $\alpha_i={}^k\alpha_i$, where the superscript denotes the k-th iteration (k=0,1,$\cdots$). Its corresponding boundary conditions (5.4) can be rewritten as $g({}^k\alpha_i)$.

For the i-th subinterval at the k-th iteration, consider the perturbed initial value problem:

$$d\beta_i/dr=0, \qquad \beta_i(t_{i-1})={}^k\alpha_i+\mu,$$

$$dy_1^{(i)}/dr=\beta_i y_2^{(i)}, \qquad y_1^{(i)}(t_{i-1})={}^kx_1^{(i)}(t_{i-1}),$$

$$dy_2^{(i)}/dr=\beta_i[-\omega^2 y_1^{(i)}-f(y_1^{(i)})+\omega^2 v^{(i)}], \quad y_2^{(i)}(t_{i-1})={}^kx_2^{(i)}(t_{i-1}),$$

$$t_{i-1}\leqq r\leqq t_i, \qquad i=1,2,\ k=0,1,\cdots, \tag{5.5}$$

where μ is a small perturbation parameter such that $0<\mu<<1$, and $v^{(1)}=Ku$ and $v^{(2)}=-Ku$. Let us define $s^{(i)}({}^k\alpha_i;\mu)$, termed as adjusting value, given by

$$s^{(i)}({}^k\alpha_i;\mu)=(1/\mu)[g^{(i)}(\beta)-g^{(i)}({}^k\alpha_i)], \quad i=1,2. \tag{5.6}$$

Using the adjusting value, we form the following iteration algorithm for the new time scale factor α_i:

$${}^{k+1}\alpha_i={}^k\alpha_i-g({}^k\alpha_i)/s^{(i)}({}^k\alpha_i;\mu), \qquad i=1,2, \qquad k=0,1,\cdots, \tag{5.7}$$

which is termed as the initial value adjusting method (IVAM). The iteration algorithm (5.7) is expected to have a nearly quadratic convergence under appropriate conditions. As for details, refer to [13-17].

6. Numerical Simulations

In the previous sections, the time optimal control problem for the rotating rigid body about the perturbed centroid O has been formulated and it has been shown that the problem can be reduced to a set of two point boundary value problems. Let us now solve the time optimal control problem.

The data for computer simulation of rotating rigid body are as follows:

(1) $a=3$, $b=2$,

(2) $\omega=0.15$,

(3) $W=\tau\varepsilon=\tau\delta=0.02$.

In the subsequent computations, the following quantities are used:

(4) the final state: $(x_1(1),\ x_2(1))=(0,\ 0)$,

(5) the integration step size for the IVAM: $h=2\times 10^{-3}$,

(6) the number of the integration steps: $m=250$ for each subinterval,

(7) the convergence criterion for the IVAM: $G=\{[g(\alpha_1)^2+g(\alpha_2)^2]/2\}^{1/2}\leqq 10^{-8}$,

(8) the perturbation parameter: $\mu=10^{-8}$,

and to compute the two point boundary value problem, the subroutine package, revised MPIVID (double precision) developed by Ojika [say, 13], has been used.

6.1. Control Without Switching

Suppose here that we try to transfer the rigid body from the given initial state to the origin without switching, and let us now solve the problem. Then, we have the following quantities:

(i) $K^*=0.34506$, $v^*=15.3361$, $E(v^*,x)=2.62096$, for $0\leqq t\leqq t_f$,

(ii) $K_0^*=0.04$, $v_0^*=1.77778$, $E(v_0^*,0)=0.01056$, for $t_f\leqq t$,

where t_f is computed to be $t_f=37.95$. The time optimal solutions without switching are shown in Fig.6.1, from which it is easily seen that the rigid body reaches from the given initial state to the origin without switching, and then it is held to the origin. The contour lines of the constant energy $E(v,x)$ for various control gains K are also shown in Fig.6.2.

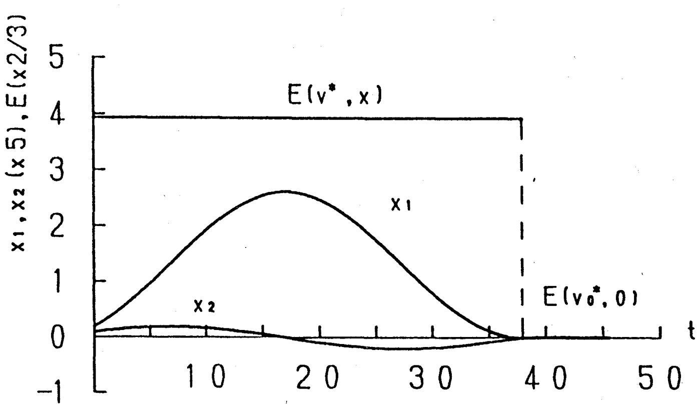


Fig.6.1. The optimal trajectories.

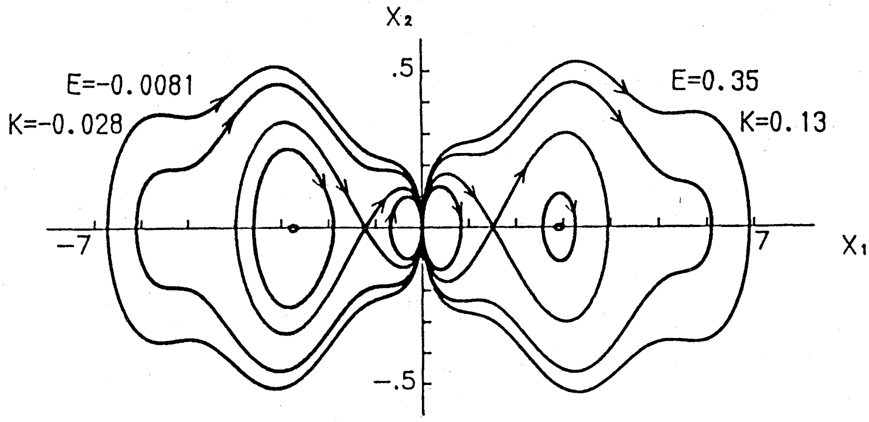


Fig.6.2. Contour lines of the constant energy $E(v,x)$.

6.2. Control With Switching

(I) Let the control gain be $K^{*}=0.25$ ($\omega^2 v^{*}=-0.25$). When $u=-1$ at $t=0$, we have $x_2^{*}(t_s)=-0.8758$, and since $x_2^{*}(t_s) < x_{20}$, the condition (i) in Control Law 4.2 is satisfied. Thus it is easily seen that the optimal control sequence for the given initial condition and the control gain is {-1, +1} and the optimal switching point is $x^{*}(t_s^{*})=(2.3803, -0.8758)$.

Starting from the initial guesses ${}^0\alpha_1=5.0$ and ${}^0\alpha_2=5.0$, the convergence behavior of G is shown in Table 6.1. The resulting optimal trajectory in the state space is given in Fig.6.1, in which the switching curve S of $(x_1^{*}(t_s), x_2^{*}(t_s))$ for various control gains K is also given.

Table 6.1 Convergence tendencies of G

iteration	G
0	0.793
1	0.142
2	0.482× 10-2
3	0.204× 10-6
4	0.160× 10-13

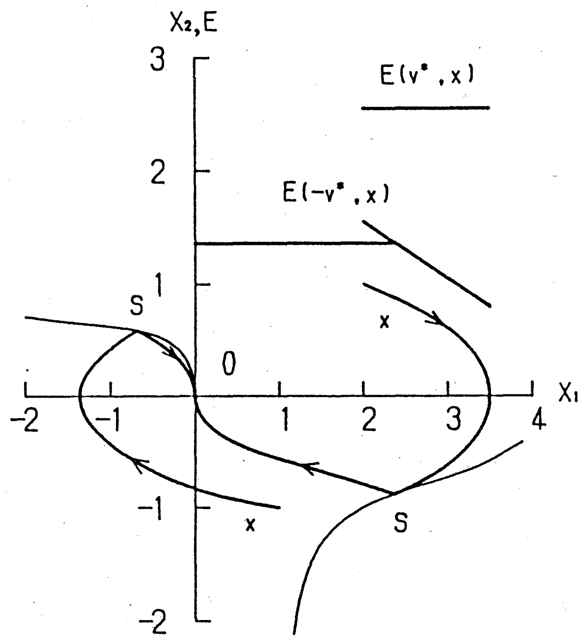


Fig. 6.3. The trajectories in the state space.

The response of the time optimal is shown in Fig.6.4, here $\alpha_1^{*}=10.497$, $\alpha_2^{*}=10.443$, $t_s^{*}=5.2487$, and $t_f^{*}=10.455$. When the rigid body reached the origin, the control is switched to the holding control $K_0^{*}u_0^{*}(t)=\omega^2 v_0^{*}=0.04$ (its corresponding holding energy is obtained to be $E(v_0^{*},0) = 0.0106$), and hence the rigid body is kept to the origin after $t \geqq t_f^{*}$ [5].

(II) Let the initial state and the control be $x(0)=(1, -1)$ and $K^{*}=0.25$ ($\omega^2 v^{*}=-0.25$), respectively. When $u=-1$ at $t=0$, we have $x_2^{*}(t_s)=0.5918$, and since $x_2^{*}(t_s) > x_{20}$, the condition (ii) in Control Law

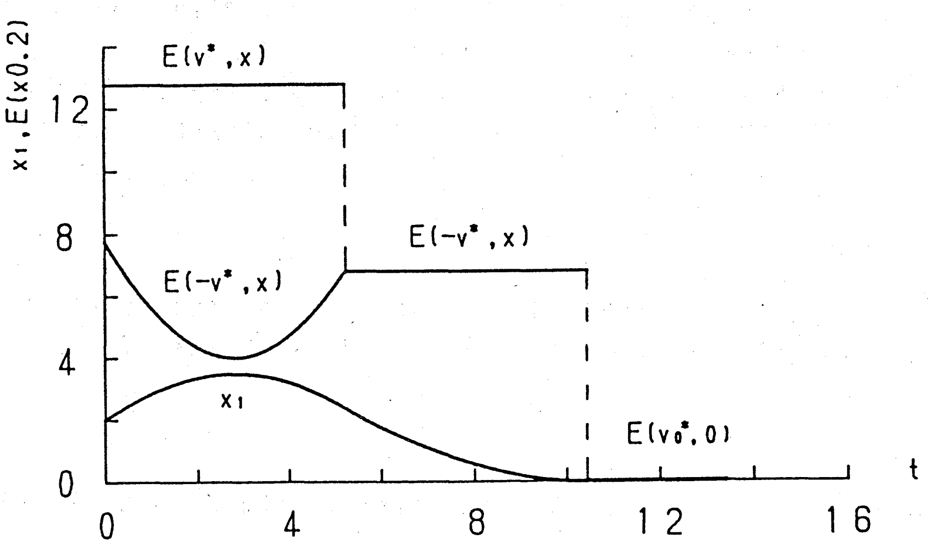


Fig. 6.4. The response of the optimal system.

4.2 is satisfied, and hence the sign of u must be changed to u=+1. Thus the control sequence in this case is {+1, -1} and the optimal switching point is $x^*(t_s^*)=(-0.6842,\ 0.5918)$. The resulting optimal trajectory in the state space is given in Fig.6.3, here $\alpha_1^*=13.385$, $\alpha_2^*=4.4450$, $t_s^*=6.6925$, and $t_f^*=8.9150$.

7. Concluding Remarks

In this paper, (1) the dynamical motion of a rigid body about the perturbed centroid, which is placed in wind, has been first formulated, (2) then, in order to control the rigid body, a control method by an air or a water jet system has been proposed, and it has been clarified that the equation has a periodic solution which is uniquely determined by the given initial state, (3) several properties of the system with a constant control are discussed from the energy point of view, and the constant control which transfers the rigid body from a given initial state to the origin and the holding control which keeps it at the origin have been given. (4) According to the max-min energy theory for the solution of time optimal control, the optimal control sequence, the switching point, and the switching surface can be easily obtained analytically without using the well-known maximum principle by Pontryagin. Moreover, (5) it is shown that if the control gain is chosen to be large enough, then the rigid body is reachable from any initial state to the origin within a switching. (6) lastly, the equation has been solved by a computer, and the effectiveness of the control algorithms are also justified quantitatively.

It is worth insisting that the max-min enery theory developed here can be easily generalized to nonlinear double integral plants without damping terms.

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